Asymptotes
(Linear asymptotes)

Our advice is to first study well the limits of functions.

Some professors like to asymptote function tests as " behavior functions in the function domain",and how they work you work the same…

Another thing, some professors do not examine horizontal asymptote as a separate, but work it in the oblique asymptotes, we'll try to explain each of asymptote in particular.

An **asymptote** of a real-valued function $y = f(x)$ is a curve which describes the behavior of f as either x or y goes to infinity. There are three types of linear asymptote:

- **vertical**
- **horizontal**
- **oblique**

Vertical asymptotes

Potential vertical asymptote is in "the interruption" of the function domain . If ,for example, point $x = \Theta$ is the interruption point, we must examine how function will "act" in the vicinity of that point, this require two limes:

$$\lim_{x\to\Theta+\varepsilon;\, when\, \varepsilon\to 0} f(x) \quad \text{and} \quad \lim_{x\to\Theta-\varepsilon;\, when\, \varepsilon\to 0} f(x)$$

If the solutions of these two limits are $+\infty$ or $-\infty$, then $x = \Theta$ is vertical asymptote.

Horizontal asymptotes

Here we are looking for two limits $\lim\limits_{x\to+\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$. If for solution we get a number, for example, Ω , then $y = \Omega$ is horizontal asymptote, and if we get $+\infty$ or $-\infty$ then we say that there is no horizontal asymptote.

Oblique asymptotes

Oblique asymptote is : **y = kx + n** where are:

$$k=\lim_{x\to\pm\infty}\frac{f(x)}{x} \qquad \text{and} \qquad n=\lim_{x\to\pm\infty}[f(x)-kx]$$

Note that **y = kx + n** is never a vertical asymptote, but can be a horizontal asymptote if n=0 (in which case it is not an oblique asymptote). Or:

If we have horizontal asymptote, there is not oblique ! Remember!

Before we begin with examples , we will remind you how to search the function domain.

The **domain** (or **replacement set**) of a given function is the set of "input" values for which the function is defined.

Domain of function:

If we have the rational function $\frac{P(x)}{Q(x)}$, then must be $Q(x)\neq 0$

If we have $\ln \otimes$, then $\otimes>0$

If we have $\sqrt{\Theta}$, then $\Theta\geq 0$

If we have $\sqrt[3]{@}$, $@\in R$

The function e^x is everywhere defined

If we have arcsin @ then is $-1\leq @\leq 1$ ….etc…

EXAMPLES

1. Find asymptote for the following functions:

a) $y=\frac{x+1}{x-1}$

b) $y=\frac{x^2-4}{x-1}$

c) $y=\frac{x^2-4}{1-x^2}$

Solution:

a) $y = \dfrac{x+1}{x-1}$

vertical asymptote

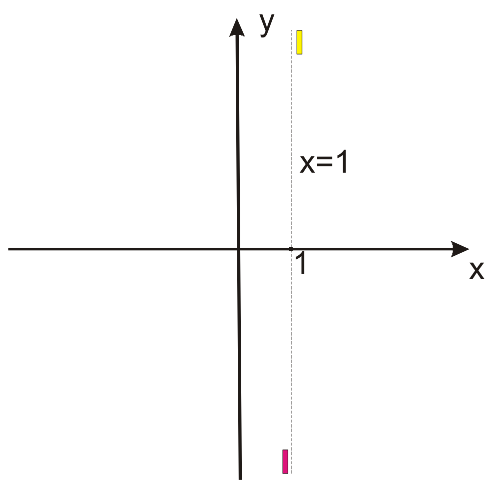
The function is defined for $x - 1 \neq 0$, that is, $x \neq 1$. This tells us that x = 1 **can be** vertical asymptote.

$$\lim_{x \to 1+\varepsilon, \varepsilon \to 0} \frac{x+1}{x-1} = \frac{1+1}{1+\varepsilon-1} = \frac{2}{+\varepsilon} = \frac{2}{+0} = +\infty$$

$$\lim_{x \to 1-\varepsilon, \varepsilon \to 0} \frac{x+1}{x-1} = \frac{1+1}{1-\varepsilon-1} = \frac{2}{-\varepsilon} = -\infty$$

x = 1 **is vertical asymptote**

What does this mean on GRAFIK? Let's take a look:



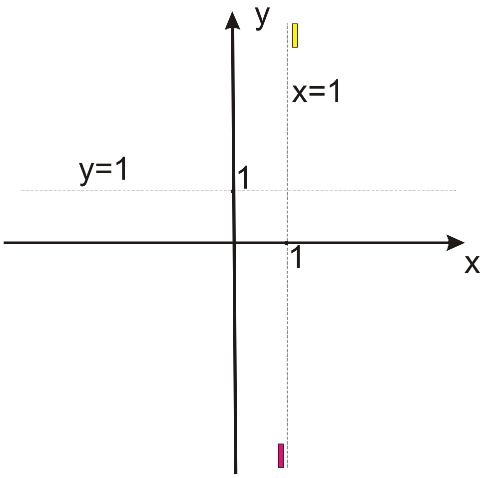
$\lim_{x \to 1+\varepsilon, \varepsilon \to 0} \frac{x+1}{x-1} = +\infty$ This is a yellow line on the graphics, which means that when x approaching 1 with the positive side (+ ε) , then function tends to ∞.

$\lim_{x \to 1-\varepsilon, \varepsilon \to 0} \frac{x+1}{x-1} = -\infty$ This is a red line on the graphics, which means that when x approaching 1 with

the negative side (-ε),then function tends to $-\infty$.

Horizontal asymptote

$\lim\limits_{x\to\pm\infty}\dfrac{x+1}{x-1}=$ 1, which means that the y = 1 is horizontal asymptote and we not have oblique asymptote. On the graphics:



b) $y=\dfrac{x^2-4}{x-1}$

vertical asymptote:

The function is defined for $x-1\neq 0 \longrightarrow x\neq 1, \longrightarrow$ x = 1 **can be** vertical asymptote.

$$\lim_{x\to 1+\varepsilon,\varepsilon\to 0}\frac{x^2-4}{x-1}=\frac{1^2-4}{1+\varepsilon-1}=\frac{-3}{+\varepsilon}=\frac{-3}{+0}=-\infty$$ (yellow line on the graphic)

$$\lim_{x\to 1-\varepsilon,\varepsilon\to 0}\frac{x^2-4}{x-1}=\frac{1^2-4}{1-\varepsilon-1}=\frac{-3}{-\varepsilon}=\frac{-3}{-0}=+\infty$$ (red line on the graphic)

horizontal asymptote:

$\lim\limits_{x\to\pm\infty}\dfrac{x^2-4}{x-1}=\pm\infty$ This tells us that there is no horizontal asymptote and we must pursuit oblique asymptote.

oblique asymptote:

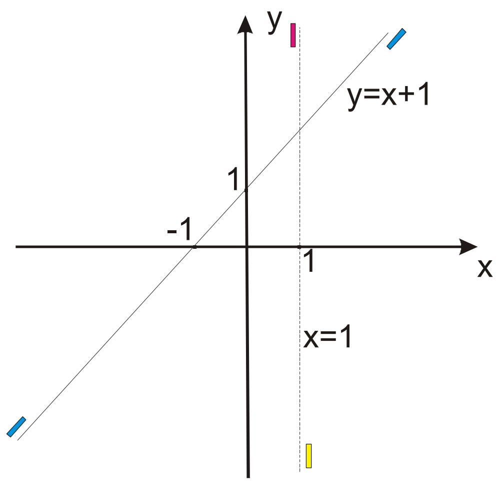
y = kx + n

k= $\lim\limits_{x\to\pm\infty}\dfrac{f(x)}{x}$ and n= $\lim\limits_{x\to\pm\infty}[f(x)-kx]$

k= $\lim\limits_{x\to\pm\infty} \dfrac{\frac{x^2-4}{x-1}}{x} = \lim\limits_{x\to\pm\infty} \dfrac{x^2-4}{x^2-x} = 1$ (see file limits of functions)

n= $\lim\limits_{x\to\pm\infty}[f(x)-kx] = \lim\limits_{x\to\pm\infty}\left[\dfrac{x^2-4}{x-1}-1x\right] = \lim\limits_{x\to\pm\infty}\left[\dfrac{x^2-4-x(x-1)}{x-1}\right] = \lim\limits_{x\to\pm\infty}\left[\dfrac{x^2-4-x^2+x}{x-1}\right] = \lim\limits_{x\to\pm\infty}\left[\dfrac{x-4}{x-1}\right]=1$

k and n change in the formula: **y = kx + n,** and get the **oblique asymptote:** **y = x + 1**



Blue line describes the behavior of function .

c) $y = \dfrac{x^2-4}{1-x^2}$

vertical asymptote:

The function is defined for $1-x^2 \neq 0 \longrightarrow (1-x)(1+x) \neq 0 \longrightarrow x \neq 1$ and $x \neq -1$

This means that we have to find two limes, for 1 and –1, with "both" sides.

$$\lim\limits_{x\to 1+\varepsilon,\varepsilon\to 0}\frac{x^2-4}{1-x^2} = \lim\limits_{x\to 1+\varepsilon,\varepsilon\to 0}\frac{x^2-4}{(1-x)(1+x)} = \frac{1^2-4}{(1-(1+\varepsilon))(1+1+\varepsilon)} = \frac{-3}{(1-1-\varepsilon)2} = \frac{-3}{(-\varepsilon)2} = +\infty$$ (blue line)

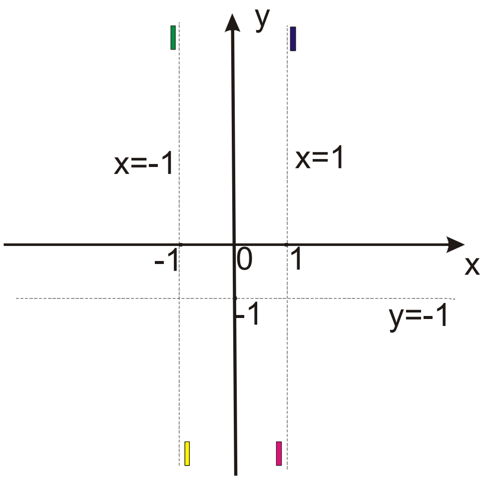
$$\lim\limits_{x\to 1-\varepsilon,\varepsilon\to 0}\frac{x^2-4}{1-x^2} = \lim\limits_{x\to 1-\varepsilon,\varepsilon\to 0}\frac{x^2-4}{(1-x)(1+x)} = \frac{1^2-4}{(1-(1-\varepsilon))(1+1-\varepsilon)} = \frac{-3}{(1-1+\varepsilon)2} = \frac{-3}{\varepsilon 2} = -\infty$$ (red line)

$$\lim\limits_{x\to -1+\varepsilon,\varepsilon\to 0}\frac{x^2-4}{1-x^2} = \lim\limits_{x\to -1+\varepsilon,\varepsilon\to 0}\frac{x^2-4}{(1-x)(1+x)} = \frac{(-1)^2-4}{(1-(-1+\varepsilon))(1+(-1+\varepsilon))} = \frac{-3}{(2-\varepsilon)\varepsilon} = \frac{-3}{2\varepsilon} = -\infty$$ (yellow line)

$$\lim\limits_{x\to -1-\varepsilon,kad\varepsilon\to 0}\frac{x^2-4}{1-x^2} = \lim\limits_{x\to -1-\varepsilon,kad\varepsilon\to 0}\frac{x^2-4}{(1-x)(1+x)} = \frac{(-1)^2-4}{(1-(-1-\varepsilon))(1+(-1-\varepsilon))} = \frac{-3}{(2+\varepsilon)(-\varepsilon)} = \frac{-3}{2(-\varepsilon)} = +\infty$$ (green line)

horizontal asymptote:

$$\lim_{x\to\pm\infty}\frac{x^2-4}{1-x^2}=-\frac{1}{1}=-1 \longrightarrow$$ y = - 1 is horizontal asymptote and there is not oblique asymptote.



2) Find asymptote for the following functions:

a) $y=e^{\frac{1}{x}}$

b) $y=xe^{\frac{1}{x}}$

Solution:

a) $y=e^{\frac{1}{x}}$

vertical asymptote:

The function is defined for $x\neq 0$,and $x=0$ is potential vertical asymptote.

$$\lim_{x\to 0+\varepsilon,\varepsilon\to 0} e^{\frac{1}{x}} = e^{\frac{1}{0+\varepsilon}} = e^{+\infty} = \infty$$ (red line on the graphic)

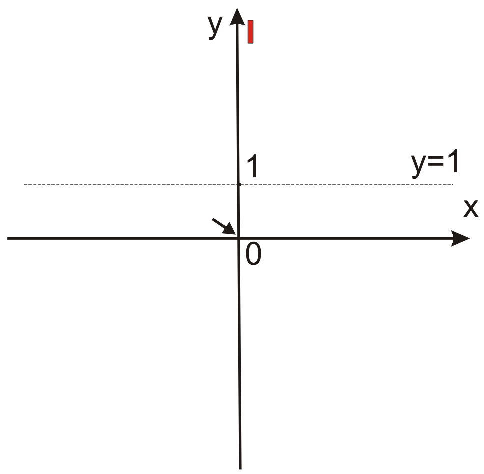
$$\lim_{x\to 0-\varepsilon,\varepsilon\to 0} e^{\frac{1}{x}} = e^{\frac{1}{0-\varepsilon}} = e^{-\infty} = 0$$ What this means ?

This means that when x tends to zero on the left, the negative side, the function tends to zero, in the graphics display with the arrow.

horizontal asymptote:

$$\lim_{x\to+\infty} e^{\frac{1}{x}} = e^{\frac{1}{+\infty}} = e^0 = 1$$

$$\lim_{x\to-\infty} e^{\frac{1}{x}} = e^{\frac{1}{-\infty}} = e^0 = 1$$ So y = 1 is a horizontal asymptote!



b) $y = xe^{\frac{1}{x}}$

vertical asymptote:

The function is defined for the $x \neq 0$; x = 0 is potential vertical asymptote.

$\lim\limits_{x\to0+\varepsilon} xe^{\frac{1}{x}} = (0+\varepsilon)e^{\frac{1}{0}} = 0\circ\infty =?$ The idea is to use **l'Hôpital's rule** (theorem), but before that have to

"remodel" function to form $\frac{0}{0}$ or $\frac{\infty}{\infty}$.

$$\lim_{x\to0+\varepsilon} xe^{\frac{1}{x}} = \lim_{x\to0+\varepsilon} \frac{e^{\frac{1}{x}}}{\frac{1}{x}} = \frac{\infty}{\infty} = \lim_{x\to0+\varepsilon} \frac{e^{\frac{1}{x}}(-\frac{1}{x^2})}{-\frac{1}{x^2}} = \lim_{x\to0+\varepsilon,} e^{\frac{1}{x}} = e^{\frac{1}{0+\varepsilon}} = e^{+\infty} = \infty$$ (yellow line)

$$\lim_{x\to0-\varepsilon} xe^{\frac{1}{x}} = (0-\varepsilon)e^{\frac{1}{-\varepsilon}} = 0\circ0 = 0$$ (arrow)

horizontal asymptote:

$$\lim_{x\to+\infty} xe^{\frac{1}{x}} = \infty \circ e^{\frac{1}{\infty}} = \infty \circ e^{0} = \infty \circ 1 = \infty$$

$$\lim_{x\to-\infty} xe^{\frac{1}{x}} = -\infty \circ e^{\frac{1}{\infty}} = -\infty \circ e^{0} = -\infty \circ 1 = -\infty$$

So, no horizontal asymptote, and we have to look for oblique asymptote.

oblique asymptote:

y = kx + n

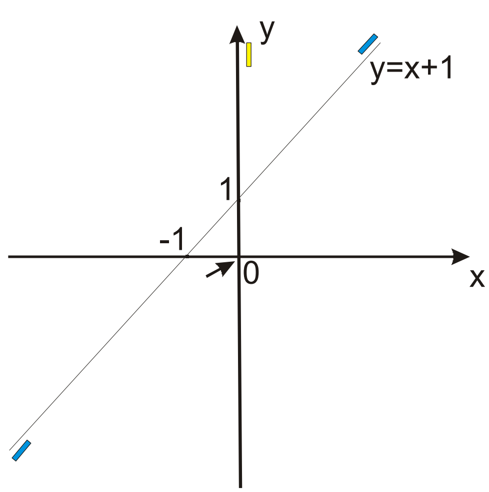
k= $\lim\limits_{x\to\pm\infty} \frac{f(x)}{x}$ and n= $\lim\limits_{x\to\pm\infty} [f(x) - kx]$

$$k = \lim_{x\to\pm\infty} \frac{f(x)}{x} = \lim_{x\to\pm\infty} \frac{xe^{\frac{1}{x}}}{x} = \lim_{x\to\pm\infty} e^{\frac{1}{x}} = e^{\frac{1}{\infty}} = e^{0} = 1$$

n= $\lim\limits_{x\to\pm\infty} [f(x) - kx] = \lim\limits_{x\to\pm\infty} [xe^{\frac{1}{x}} - 1x] = \lim\limits_{x\to\pm\infty} x[e^{\frac{1}{x}} - 1] = \lim\limits_{x\to\pm\infty} \frac{e^{\frac{1}{x}} - 1}{\frac{1}{x}} = \frac{0}{0} =$ **l'Hôpital's rule,** derivatives…

$$n = \lim_{x\to\pm\infty} \frac{e^{\frac{1}{x}}(-\frac{1}{x^2})}{-\frac{1}{x^2}} = \lim_{x\to\pm\infty} e^{\frac{1}{x}} = e^{\frac{1}{\infty}} = e^{0} = 1$$

We have oblique asymptote: y = x +1



3. Find asymptote for function $y = \frac{x-2}{\sqrt{x^2+4}}$

Solution:

The function is defined everywhere,because $x^2 + 4 > 0$,for every x, and it tells us that it has no vertical asymptote.

horizontal asymptote:

$$\lim_{x\to\pm\infty} \frac{x-2}{\sqrt{x^2+4}} = \lim_{x\to\pm\infty} \frac{x-2}{\sqrt{x^2(1+\frac{4}{x^2})}} = \lim_{x\to\pm\infty} \frac{x-2}{|x|\sqrt{(1+\frac{4}{x^2})}}$$

Look out! As we get below the absolute value, we have to separate limits for the + ∞ and - ∞

$$\lim_{x\to+\infty} \frac{x-2}{x\sqrt{(1+\frac{4}{x^2})}} = 1$$

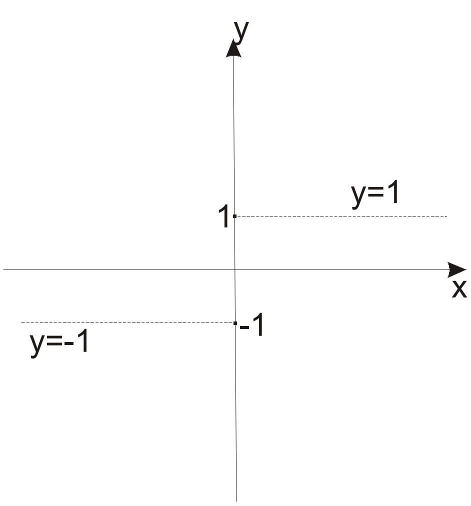
$$\lim_{x\to-\infty} \frac{x-2}{-x\sqrt{(1+\frac{4}{x^2})}} = -1$$

Very unusual situation that still occurs in root functions:

If x tends $+\infty$ horizontal asymptote is y = 1

If x tends to $-\infty$ horizontal asymptote is y = -1

The picture would look like this:



4. **Find asymptote for function** $y = \ln\dfrac{x-2}{x+1}$

Solution:

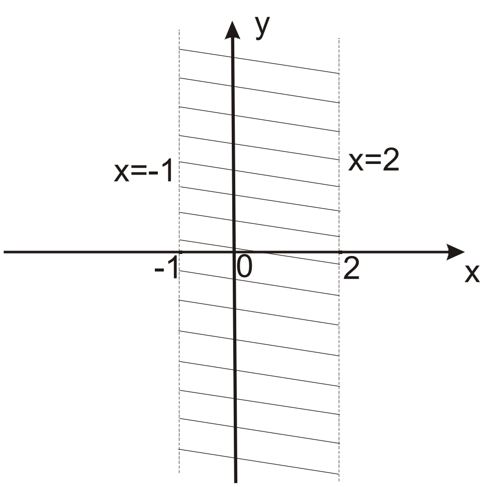
As always, first we must examine the function domain.

$\dfrac{x-2}{x+1} > 0$ It is best to go through the tables (see file inequalities)

	$-\infty$ -1	-1 2	2 $+\infty$
x-2	-	-	+
x+1	-	+	+
$\dfrac{x-2}{x+1}$	+	-	+

This therefore means that the function is defined $\forall x \in (-\infty,-1) \cup (2,\infty)$

Function does not exist between –1 and 2.



That means that we will ask for x = 2 limit only on the right side, and x = -1 only on the left side!

$$\lim_{x\to 2+\varepsilon} \ln\frac{x-2}{x+1} = \ln\frac{2+\varepsilon-2}{2+1} = \ln 0 = -\infty \quad \text{(red line)}$$

$$\lim_{x\to -1-\varepsilon} \ln\frac{x-2}{x+1} = \ln\frac{-1-2}{-1-\varepsilon+1} = \ln\frac{-3}{-\varepsilon} = \ln\infty = \infty \quad \text{(green line)}$$

horizontal asymptote:

$$\lim_{x\to\pm\infty} \ln\frac{x-2}{x+1} = \ln \lim_{x\to\pm\infty} \frac{x-2}{x+1} = \ln 1 = 0$$

So: y = 0 is a horizontal asymptote. (Blue line)